APPROXIMATE COMPUTATION OF OPTIMAL CONTROL BY THE DIRECT METHOD

(O PRIBLIZHENOM VYCHISLENII OPTIMAL'NOGO UPRAVLENIIA PRIAMYM METODOM)

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A description is given of an approximate method for computing the optimal (in regard to speed) trajectory [1] in a linear control system. The optimal problem is replaced by an auxiliary "smoothed out" problem which is treated by the usual methods of the variational calculus.

1. Let us consider a control system described by the differential equation

$$\frac{dx}{dt} = Ax + bu \qquad (1.1)$$

where x is an n-dimensional vector of the phase coordinate system, A is an $n \times n$ matrix, b is an n-vector which characterizes the structure of the system, and $u(t)$ is a scalar function that describes the control action. The admissible controls are subjected to the condition

$$|u(t)| \leqslant 1 \qquad (1.2)$$

The optimal problem [1-3] is formulated in the following way.

Problem A. Under the given initial conditions $x = x_0$, $t = t_0 = 0$ it is required to determine an admissible function $u^\circ(t)$ (optimal control) such that the trajectory $x^\circ(t) = x(x_0, t, u^\circ)$ of the system (1) generated by the control $u^\circ(t)$ will arrive at the point $x = 0$ in the shortest time T°. The existence of the solution of this problem is assured by the maximum principle [2], while the optimal trajectory $x^\circ(t)$ and the auxiliary vector functions $\psi^\circ(t)$ of the "momenta" $\psi_i^\circ(t)$ are the solutions of some Hamiltonian system under the initial conditions $\psi_i^\circ(0) = c_{i0}$ $(i = 1, \ldots, n)$, which guarantee that the trajectory $x^\circ(t)$ will pass through the point $x = 0$. The difficulty in the concrete computation of $u^\circ(t)$ and $x^\circ(t)$ arises, in particular, out of the determination of the constants c_{i0}. In the present note there is described an approximate method for computing $u^0(t)$ and $x^0(t)$. The problem is thus reduced to the classical variational problem that is solved by the direct method.

Along with problem A, we shall also consider auxiliary problems. Let $U_\epsilon[x, t]$ be a function which is continuously differentiable with respect to x_i, and satisfies the conditions

$$0 \leqslant U_\varepsilon \leqslant 1 \quad \text{for all } t, x; \quad U_\varepsilon[0, t] = 0; \quad U_\varepsilon[x, t] = 0 \quad \text{if } t > \tau_\varepsilon \tag{1.3}$$

$$U_\varepsilon[x, t] \geqslant q(\varepsilon) \quad \text{if } \|x\| \geqslant \varepsilon > 0, \ 0 \leqslant t \leqslant \theta_\varepsilon \tag{1.4}$$

$$(\|x\| = (x_1^2 + \ldots + x_n^2)^{1/2})$$

$$\lim q(\varepsilon) = 1, \quad \lim \theta_\varepsilon = \infty, \quad \lim \tau_\varepsilon = \infty \quad \text{if } \varepsilon \to 0 \tag{1.5}$$

Problem A_ϵ. For given $x = x_0$, $t_0 = 0$, it is required to determine an ϵ admissible control $u_\epsilon(t)$ such that

$$T_\varepsilon^\circ = \int_0^\infty U_\varepsilon[x(x_0, t, u_\varepsilon^\circ), t]\, dt = \min \tag{1.6}$$

Problem $A_{\epsilon k}$. For given $x = x_0$, $t_0 = 0$ it is required to determine a control $u_{\epsilon k}^\circ(t)$ such that

$$T_{\varepsilon k}^\circ = \int_0^\infty (U_\varepsilon[x(x_0, t, u_{\varepsilon k}^\circ), t] + [u_{\varepsilon k}^\circ(t)]^{2k})\, dt - T^\circ = \min \tag{1.7}$$

where k is a positive integer and the function $u_{\epsilon k}^\circ$ is restricted by the condition (1.2).

In the sequel it will be assumed that the point x_0 lies in the region G_0 in which there exists an admissible control at the point $x = 0$ for a finite time $T(x_0)$. In particular, if the condition

$$\sum_{l=0}^{n-1} \lambda_l A^l b \neq 0 \quad \text{when} \quad \sum_{l=0}^{n-1} \lambda_l^2 \neq 0 \tag{1.8}$$

is satisfied, then G_0 will be an open region containing the point $x = 0$. If in this case the characteristic values of the matrix A have negative real parts, then the region G_0 coincides [2-5] with the entire space $\{x\}$.

2. The following result can serve as a basis for the replacement of problem A by the problems A_ϵ and $A_{\epsilon k}$ for small $\epsilon > 0$ and for large k.

Theorem 2.1. The following equations hold:

$$\lim T_\varepsilon^\circ = T^\circ \qquad \text{as} \quad \varepsilon \to 0 \tag{2.1}$$

$$\lim T_{\varepsilon k}^\circ = T^\circ \qquad \text{as} \quad \varepsilon \to 0,\ k \to \infty \tag{2.2}$$

where the function $u_\epsilon^\circ(t)$ and $u_{\epsilon k}^\circ(t)$ converge in the mean to $u^\circ(t)$, as $\epsilon \to 0$, $k \to \infty$, on the interval $[0, T^\circ]$ (in L_2).

Proof. This theorem can be proved on the basis of the results given in [3-6]. We give, however, a detailed proof based on the arguments presented in [2-7].

The following inequalities are obviously true

$$T_\varepsilon^\circ < T^\circ, \qquad T_{\varepsilon k}^\circ < T^\circ \tag{2.3}$$

Let us consider briefly the question of the existence of the optimal control $u^\circ(t)$. Since there exists an admissible control $u(t)$ from x_0 at the point $x = 0$ during time $T(x_0)$, the next equation is valid

$$-x_0 = \int_0^{T(x_0)} F^{-1}(\tau)\, b\, u(\tau)\, d\tau \tag{2.4}$$

where $F^{-1}(\tau) = \| F_{ij}^{-1} \|_1^n$ is a matrix which is the inverse of the fundamental matrix of the solution of the system $dx/dt = Ax$.

This means that there exists a linear functional [8]

$$f[h] = \int_0^{T(x_0)} h(\tau)\, u(\tau)\, d\tau \qquad \left(h \in L(0,\ T(x_0)) \right) \tag{2.5}$$

such that

$$f[h_i] = -x_{i0}, \qquad h_i(\tau) = \sum_{j=1}^n F_{ij}^{-1}(\tau)\, b_j \qquad (i = 1,\ldots,n)$$

$$\|f\| = \sup\left(|u(\tau)| \leqslant 1 \quad \text{if} \quad 0 \leqslant \tau \leqslant T(x_0)\right) \tag{2.6}$$

Suppose that among the furctions $h_i(\tau)$, $0 < \tau < T(x_0)$ there are exactly m ($m < n$) linear independent ones, and let us assume, for the sake of definiteness, that these are the first m functions h_i, while $h_j = \lambda_{ji} h_1 + \ldots + \lambda_{jm} h_m$ ($j = m + 1, \ldots, n$). Under these assumptions we have, by (2.5), $x_{j0} = \lambda_{j1} x_{10} + \ldots + \lambda_{jm} x_{m0}$ ($j = m + 1, \ldots, n$) and if the control u° is to pass through the point $x = 0$ when $t = T^\circ < T(x_0)$, it is sufficient that the following conditions be satisfied:

$$f^\circ[h_i] = -x_{i0} \qquad (i = 1,\ldots,m) \qquad \|f^\circ\| \leqslant 1 \tag{2.7}$$

where the functional f° is determined by the function u°:

$$f^\circ[h] = \int_0^{T^\circ} h(\tau)\, u^\circ(\tau)\, d\tau \qquad (2.8)$$

One can construct the functional f° (or, what is the same thing, the function $u^\circ(\tau)$ [8]) satisfying the conditions (2.7) if, and only if [7]

$$\min\left[\int_0^{T^\circ}\left|\sum_{i=1}^m \lambda_i h_i(\tau)\right| d\tau\right] \geqslant 1 \qquad \left(\sum_{i=1}^m \lambda_i x_{i0} = -1\right) \qquad (2.9)$$

where, in our case, the function $u^\circ(\tau)$ (for the smallest T°) is uniquely determined by the condition [3,7]

$$u^\circ(\tau) = \operatorname{sign}\left[\sum_{i=1}^m \lambda_i^\circ h_i(\tau)\right] \qquad (2.10)$$

Here, λ_i° is the solution of the problem (2.9). We have thus obtained, as a byproduct, the verification that $u^\circ(\tau)$ has the form (2.10) and that the point $x = 0$ can be reached only from the points $x_{j0} = \lambda_{j1}x_{10} + \ldots + \lambda_{jm}x_{m0}$ $(j = m + 1, \ldots, n)$. We note here that, since the functions $h_i(\tau)$ are constructed from the solutions $F_{ij}^{-1}(\tau)$ of a stationary linear system of differential equations, the quantity $\lambda_1^\circ h_1(\tau) + \ldots + \lambda_m^\circ h_m(\tau)$ reduces to zero only at isolated values of τ. Let us now assume the opposite, that the condition (2.1) (or (2.2)) is not satisfied. On the basis of (1.6), (1.7), (1.3)-(1.5) and (2.3) we can conclude that there exists a sequence

$$u_l = u_{\varepsilon_l}(t) = (\text{or } u_l^* = u_{\varepsilon_l k_l}(t) = [\operatorname{sign} u_{\varepsilon_l k_l}(t)](\sup[|u_{\varepsilon_l k_l}(t)|, 1]))$$

such that the trajectory $x(x_0, t, u_l)$ (or $x(x_0, t, u_l^*)$ falls on the surfaces $\|x\| = \epsilon_l$ during the time T_l (or T_l^*) while

$$\lim \varepsilon_l = 0, \qquad \lim T_l = T_0 < T^\circ \quad (\text{or } \lim T_l^* = T_0^* < T^\circ) \qquad \text{as } l \to \infty$$

One can select, from this sequence, a subsequence which converges weakly in $L_2(0, T_0)$ to the function $u_0(t)$ (or to the function $u_0^*(t)$, respectively). In the formula for the solution of the system (1.1)

$$x(t) = F(t)\,x_0 + \int_0^t F(t)\,F^{-1}(\tau)\,bu(\tau)\,d\tau \qquad (2.11)$$

for this subsequence, it is permissible to interchange the order of taking the limit and integrating. The control functions (which are measurable) $u_0(t)$ (or $u_0^*(t)$) will make the trajectories of the system (1.1) pass through the point $x = 0$ when $t = T_0 < T^\circ$ (or $t = T_0^* < T^\circ$). This is, however, impossible because $u^\circ(t)$ is an optimal control. In accordance with the considerations given at the beginning of the proof of the theorem, the minimum is reached on the function $u^\circ(t)$ of the type (2.10) not only relative to the admissible class of piece-wise continuous functions but also relative to a broader class of measurable, bounded functions. The derived contradiction proves the validity of Equations (2.1) and (2.2).

The correctness of the second conclusion of the theorem under conditions (2.1) and (2.2) can be established in the same way by starting with the weak compactness of the functions $u_\epsilon(t)$ (or of $u_{\epsilon k}^*(t) = [\operatorname{sign} u_{\epsilon k}(t)] \sup (|u_{\epsilon k}(t)|, 1)$) when $0 \leqslant t \leqslant T^\circ$, and on the basis of the uniqueness of the optimal control $u^\circ(t)$ determined by (2.10).

3. In this section we derive necessary conditions for the optimality of the control $u_\epsilon^\circ(t)$ and the trajectory $x_\epsilon^\circ(t)$ of problem A.

Theorem 3.1. The optimal control $u_\epsilon^\circ(t)$ for problem A satisfies the condition

$$u_\varepsilon^\circ(t)\ (\psi^\circ(t)\cdot b) = \max \tag{3.1}$$

where the vector function $\psi^\circ(t)$ is a particular solution of the system

$$\frac{d\psi}{dt} = -A'\psi + \eta(t) \qquad \left(\eta_i(t) = \frac{\partial U_\varepsilon[x_\varepsilon^\circ(t), t]}{\partial x_i}\right) \tag{3.2}$$

(A' is the transposed matrix of A, $\eta(t)$ is a vector function).

Note 3.1. If, in particular, $U[x, 1] = 1$ when $\|x\| > \epsilon$, $t < \theta_\epsilon$, then the vector function $\psi^\circ(t)$, which by (3.1) determines an optimal control $u_\epsilon^\circ(t)$, will be a particular solution of the system

$$\frac{d\psi}{dt} = -A'\psi \qquad (\text{if} \quad \|x\| > \varepsilon) \tag{3.3}$$

This coincides with the conditions of the maximum principle [2] for problem A which are connected with conditions (3.1) and (3.2) by theorem 2.1.

Proof of Theorem 3.1. Problem A_ϵ is an ordinary variational problem. Since the solution of the system (1.1) is given by Equation (2.11), the variation δT_ϵ of the functional (1.6), which corresponds to the variation $\delta u_\epsilon(t)$, has the form

$$\delta T_\varepsilon = \int_0^\infty \left[\sum_{i=1}^n \frac{\partial U_\varepsilon[x_\varepsilon^\circ(t), t]}{\partial x_i} \int_0^t \left[\sum_{j=1,\ e=1,}^n F_{ij}(t) F^{-1}{}_{je}(\tau)\, b_e \delta u(\tau)\right] d\tau\right] dt$$

If the variation $\delta u(\tau)$ is zero everywhere outside a neighborhood of the point $\tau = \tau^*$, then

$$\operatorname{sign}\delta T_\varepsilon = -f(\tau^*)\operatorname{sign}\delta u(\tau^*) =$$
$$= \operatorname{sign}\delta u(\tau^*)\int\limits_{\tau^*}^{\infty}\left[\sum_{i=1}^{n}\frac{\partial U_\varepsilon}{\partial x_i}\left[\sum_{j=1,\,e=1}^{n} F_{ij}(t)\,F_{je}^{-1}(\tau^*)\,b_e\right]\right]dt \qquad (3.4)$$

The matrix $F^{-1}(t)$ is the transposed fundamental matrix of the solution of the system (3.3). By differentiating the integral (3.4) with respect to the parameter τ^* we can, therefore, verify that the function $f(t)$ can be considered as the scalar product of the vector b by some particular solution $\psi^\circ(t)$ of the system (3.2). Because of the minimal nature of T_ϵ°, the variation δT_ϵ cannot be negative, which, in view of (3.4), proves the theorem.

Note 3.2. The use of (3.1) and (3.2) is difficult in practical computations of trajectories, because no rule is given for the determination of the initial conditions of the solution $\psi^\circ(t)$. However, the consideration of the smoothed-out problem A_ϵ reveals the possibility of treating the optimal problem A as an ordinary variational problem. Furthermore, from this point of view one can easily see the connection between the methods of solving the optimal problems with the classical variational methods of mechanics. Hereby, the function $U_\epsilon[x,\ t]$ plays formally the same role as the terms which corresponds to the potential energy in Hamilton's function of mechanical problems. Incidentally, we mention the fact that by this analogy the general Liapunov function $v^\circ(x,\ t)$, defined in the work [5], corresponds to the action S of a mechanical system in the theory of the Hamilton-Jacobi equation. Rozonoer [9] has considered the analogy between the variational principles of mechanics and the solution of optimal problems from a somewhat different aspect to ours.

4. In this section we describe the direct method of solution of the problem A_ϵ.

We choose for our function $U_\epsilon[x,\ t]$ the function

$$U_\varepsilon = 1 - \exp(-x^2/2\varepsilon) \quad \text{if} \quad t\in[0,\ \tau_\varepsilon], \qquad U_\varepsilon = 0 \quad \text{if} \quad t>\tau_\varepsilon$$

where τ_ϵ is a sufficiently large number. In other words, we replace the integral (1.7) by an integral of the function $U_\epsilon = 1 - \exp(-x^2/2\epsilon)$ over the finite interval $[0,\ \tau_\epsilon]$. The problem $A_{\epsilon k}$ can be solved by the direct method.

$$u(t) = a_1\sin(t/\tau_\varepsilon) + \ldots + a_l\sin(lt/\tau_\varepsilon) \qquad (4.1)$$

The substitution of $u(t)$ into Formulas (2.11) and (1.7) reduces the problem to the form

$$\min F(a_1,\ldots,a_l) = \min \Big[\int_0^{\tau_\varepsilon} (U_\varepsilon [x(x_0, t, u), t] + u^{2k}(t))\, dt \Big]$$
$$\text{if } (-\infty < a_i < \infty, i = 1,\ldots,l \tag{4.2}$$

If the number $\epsilon > 0$ is sufficiently small, and the numbers k and l are sufficiently large, then the control $u'(t) = a_1^\circ \sin(t/\tau_\epsilon) + \ldots + a_l^\circ(lt/\tau_\epsilon)$, where a_i° is a solution of the problem (4.2), will differ (in the mean) by an arbitrarily small amount from the optimal control $u^0(t)$, and the trajectory $x(x_0, t, u')$ will be arbitrarily close to $x^0(t)$ at every moment t of the transition process. Indeed, the minimum of the function $F(a_1, \ldots, a)$ is obviously not less than $T_{\epsilon k}^\circ$. At the same time, the function $u^0(t)$ (for large enough l) can be approximated (in the mean) arbitrarily closely by the polynomial (4.1). Hence, for small $\epsilon > 0$ and large k and l the quantity min $F(a_1, \ldots, a_l)$ will differ from $T_{\epsilon k}^\circ$ by an arbitrarily small amount.

Thus, in accordance with Theorem 2.1, with $\epsilon \to 0$, $l \to \infty$, the solution (4.1) of the problem (4.2) has to converge in the mean to $u^0(t)$, which proves our assertion.

The minimum (4.2) can be found by the method of fastest descent. If the coefficients a_i are considered as functions of the parameter θ, then, in order to find the minimizing values a_i°, one has to integrate (numerically) the system of equations

$$\frac{da_i(\vartheta)}{d\vartheta} = -\int_0^{\tau_\varepsilon} \Big(\sum_j^n \frac{\partial U_\varepsilon}{\partial x} \Big[\int_0^t \sum_{q,s}^n F_{jq} F_{qs}^{-1}\, b_s \sin \frac{i\tau}{\tau_\varepsilon}\, d\tau \Big] + 2ku^{2k-1}(t) \sin \frac{it}{\tau_\varepsilon} \Big)\, dt$$
$$F_{jq} = F_{jq}(t),\ F_{qs}^{-1} = F_{qs}^{-1}(\tau) \qquad (i = 1, \ldots,) \tag{4.3}$$

with $\theta \geqslant 0$, and with the initial conditions $a_{i0} = a_i(0)$ which lie on the sphere of attraction of the singular point $a_i = a_i^\circ$ of the system (4.3). The process of fastest descent along the trajectory (4.3) can be accomplished on an optimizer, a system of extremal control [10]. Hereby the restriction on $u(t)$, which is produced by the term u^{2k}, can be replaced by another one, which is more convenient in modelling.

Note 4.1. The described method for computing the optimal trajectory by the method of smoothing out the problem and the application of direct methods, can be applied also in the case of a nonlinear, nonstationary control system.

4.2. It should be mentioned that the described method of solution makes it possible to determine only the individual optimal control $u^{\circ}(t)$ for previously-given initial conditions. This method is not very effective in applications to a problem of synthesis, i.e. to the problem of determining the optimal control u° in the form of a function $u^{\circ}(x)$ of phase coordinates x.

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